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THE STABILITY OF EQUILIBRIUM OF NON-STATIONARY SYSTEMS†

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The effect of external non-stationary perturbations on the asymptotic properties of solutions of systems of non-linear differential equations is investigated. A method is proposed for constructing Lyapunov functions, aimed at establishing conditions under which the perturbations will not destroy the asymptotic stability, instability or boundedness of solutions of a system. It is shown that the order of the perturbations may be less than that of the right-hand sides of the unperturbed equations.

1. Consider the system of differential equations

$$\dot{x}_s = f_s(\mathbf{X}) \tag{1.1}$$

(throughout this paper, $s = 1, \ldots, n$).

Together with system (1.1), we will consider the perturbed system

$$\dot{x}_s = f_s(\mathbf{X}) + \sum_{j=1}^{k} b_{sj}(t) h_j(\mathbf{X}) \tag{1.2}$$

where $f_s(\mathbf{X})$ are homogeneous functions of order μ, $h_j(\mathbf{X})$ are homogeneous functions of order σ, where μ and σ are rational numbers with odd denominators, and the functions $b_{sj}(t)$ are continuous and bounded for $t \geqslant 0$ together with their integrals

$$\int_0^t b_{sj}(\tau) d\tau$$

We shall assume that $f_s(\mathbf{X})$ are twice continuously differentiable and $h_j(\mathbf{X})$ are continuously differentiable functions, $\mu > 1$, $\sigma > 1$.

Our problem is, under what conditions does asymptotic stability (or instability) of the solution $\mathbf{X} = \mathbf{0}$ of system (1.1) entail the similar property of the trivial solution of the perturbed system?

2. Stability theorems based on non-linear approximation were first established by Malkin and Krasovskii [1, 2]. They assumed, however, that the order of the perturbations was higher than that of the right-hand sides of system (1.1), i.e. $\sigma > \mu$. When $\sigma = \mu$, however, perturbations of the above form may destroy the asymptotic stability of linear systems [3]. In other words, if a system

$$\dot{x}_s = \sum_{j=1}^{n} a_{sj} x_j$$

where a_{sj} are constant coefficients, is asymptotically stable, this does not necessarily imply that the same is true of the perturbed system

$$\dot{x}_s = \sum_{j=1}^{n} (a_{sj} + b_{sj}(t)) x_j$$

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We shall show that for non-linear systems asymptotic stability of the trivial solution may also be preserved when $\sigma \leqslant \mu$.

Theorem 1. Let the trivial solution of system (1.1) be asymptotically stable, and assume that

$$2\sigma > \mu + 1 \tag{2.1}$$

Then the trivial solution of system (1.2) is also asymptotically stable.

Proof. It has been shown [4] that if the trivial solution of system (1.1) is asymptotically stable, then positive definite functions $V(\mathbf{X})$ and $W(\mathbf{X})$ exist, homogeneous of orders m and $m + \mu - 1$, respectively, such that

$$\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} f_s(\mathbf{X}) = -W(\mathbf{X})$$

and moreover the function $V(\mathbf{X})$ is twice continuously differentiable.

Construct a Lyapunov function for system (1.2), in the form

$$V_1 = V - \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} \sum_{j=1}^{k} \int_0^t b_{sj}(\tau) d\tau h_j(\mathbf{X}) \tag{2.2}$$

It can be verified that $V_1(t, \mathbf{X})$ satisfies all the conditions of Lyapunov's asymptotic stability theorem [5].

This method of constructing Lyapunov functions for non-linear non-stationary systems may also be used to obtain sufficient conditions for instability.

Suppose that the trivial solution of system (1.1) is unstable, and that a function $V(\mathbf{X})$ exists satisfying the conditions of Lyapunov's first instability theorem [5]. Let us assume that $V(\mathbf{X})$ is a twice continuously differentiable homogeneous function. Conditions for the existence of functions of this kind were obtained in [2, 6].

Theorem 2. If inequality (2.1) is true, the trivial solution of system (1.2) is unstable.

The proof is analogous to that of Theorem 1.

Let us now investigate the conditions for system (1.2) to be dissipative.

Definition [7]. System (1.2) is said to be uniformly dissipative if a number $R > 0$ exists such that, for any $H > 0$, one can find a number $T > 0$ for which, whenever $t_0 \geqslant 0$, $t \geqslant t_0 + T$

$$\| \mathbf{X}(t, \mathbf{X}_0, t_0) \| < R$$

provided that $\| \mathbf{X}_0 \| \leqslant H$.

We shall assume that $f_s(\mathbf{X})$ and $h_j(\mathbf{X})$ are continuous homogeneous functions, $\mu > 0$, $\sigma > 0$. Suppose that the trivial solution of system (1.1) is asymptotically stable, and that a positive definite continuously differentiable homogeneous function $V(\mathbf{X})$ exists whose derivative along trajectories of system (1.1) is negative definite. Using this function, one can establish [7] that, if $\sigma < \mu$, system (1.2) will be uniformly dissipative. The construction of a Lyapunov function for system (1.2) in the form (2.2) enables one to improve the dissipativity conditions in the case when $0 < \mu < 1$.

Theorem 3. If the functions

$$\frac{\partial V}{\partial x_s} h_j(\mathbf{X}), \quad j = 1, \ldots, k$$

are continuously differentiable and, moreover

$$0 < \mu < 1, \quad 2\sigma < \mu + 1$$

then system (1.2) is uniformly dissipative.

Remarks. 1. The method proposed here for constructing Lyapunov functions may also be used to investigate certain types of non-linear systems in which the right-hand sides are not homogeneous functions of the same order.

2. If the functions $b_{sj}(t)$ satisfy certain additional conditions, the assertions of Theorems 1–3 may be strengthened, using a slightly modified construction of the function $V_1(t, \mathbf{X})$ [8].

3. We will now consider a few examples illustrating the application of the above theorems.

Example 1. Suppose we are given a system of equations

$$\dot{x}_s = \partial U(\mathbf{X}) / \partial x_s \tag{3.1}$$

where $U(\mathbf{X})$ is a continuously differentiable negative definite homogeneous function of order μ, $\mu > 1$.

We know [5] that the trivial solution of system (3.1) is asymptotically stable, and a suitable Lyapunov function is

$$V(\mathbf{X}) = \frac{1}{2}\sum_{s=1}^{n} x_s^2 \tag{3.2}$$

Consider the perturbed system

$$\dot{x}_s = \frac{\partial}{\partial x_s}\left(U(\mathbf{X}) + \sum_{j=1}^{k} b_j(t) U_j(\mathbf{X}) \right) \tag{3.3}$$

where $U_j(\mathbf{X})$ are continuously differentiable functions, homogeneous of order σ, $\sigma > 1$, and the functions $b_j(t)$ are continuous and bounded for $t \geqslant 0$ together with the integrals

$$\int_0^t b_j(\tau) d\tau \tag{3.4}$$

Using the function (3.2), we see that if $\sigma > \mu$, the trivial solution of system (3.3) is asymptotically stable, but if $\sigma < \mu$, system (3.3) is uniformly dissipative.

The function V_1 constructed according to formula (2.2) is

$$V_1 = V(\mathbf{X}) - \sigma \sum_{j=1}^{k} \int_0^t b_j(\tau) d\tau U_j(\mathbf{X})$$

Applying Theorems 1 and 3 we can improve the conditions for asymptotic stability if $\mu > 2$: $2\sigma > \mu + 2$; but for $1 < \mu < 2$ the dissipativity conditions are improved: $2\sigma < \mu + 2$.

Example 2. Consider a mechanical system whose performance is governed by the equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = \frac{\partial U}{\partial q_s}$$

where the function

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} \tag{3.5}$$

corresponds to the kinetic energy of the system; the matrix $\mathbf{A}(\mathbf{q})$ is continuously differentiable and the quadratic form

$$\dot{\mathbf{q}}^T \mathbf{A}(\mathbf{0})\dot{\mathbf{q}}$$

is positive definite.

Let U be expressed in the form

$$U(\mathbf{q}) = W(\mathbf{q}) + R(\mathbf{q})$$

where $W(\mathbf{q})$ is a continuously differentiable positive definite homogeneous function of order μ, $\mu > 2$, and $R(\mathbf{q})$ is continuously differentiable and has the property

$$\| \mathbf{q} \|^{1-\mu} \partial R / \partial \mathbf{q} \to \mathbf{0} \quad \text{as} \quad \mathbf{q} \to \mathbf{0}.$$

By Lyapunov's theorem on the instability of equilibrium (see [5]), the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ is unstable, and the function

$$V=\sum_{s=1}^{n} q_s \frac{\partial T}{\partial \dot{q}_s}$$

satisfies the conditions of Lyapunov's first instability theorem.

Let us assume, moreover, that

$$U(\mathbf{q},t)=W(\mathbf{q})+\sum_{j=1}^{k} b_j(t)U_j(\mathbf{q})+R(\mathbf{q})$$

We will assume that the perturbations have the properties indicated in Example 1.

Taking a Lyapunov function

$$V_1=\sum_{s=1}^{n} q_s \frac{\partial T}{\partial \dot{q}_s}-\sigma\sum_{j=1}^{k}\int_0^t b_j(\tau)d\tau U_j(\mathbf{q})$$

for the perturbed system, we see that if $2\sigma > \mu + 2$, the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ will also be unstable.

Example 3. Suppose the motion of a mechanical system is governed by the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}-\frac{\partial T}{\partial q_s}=-\frac{\partial P}{\partial q_s}+Q_s(t,\dot{\mathbf{q}}) \tag{3.6}$$

where the kinetic energy $T(\mathbf{q}, \dot{\mathbf{q}})$, as in Example 2, is the quadratic form (3.5) and $P(\mathbf{q})$ is a positive definite twice continuously differentiable homogeneous function of order λ, $\lambda > 2$.

Let us assume that the generalized forces are

$$Q_s=\partial W(\dot{\mathbf{q}})/\partial \dot{q}_s$$

where $W(\dot{\mathbf{q}})$ is a continuously differentiable negative-definite homogeneous function of order, μ, $\mu > 2$. Thus, the generalized forces are dissipative and the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (3.6) is asymptotically stable [7].

Consider the Lyapunov function [9]

$$V=T+P+\alpha\sum_{s=1}^{n}\frac{\partial T}{\partial \dot{q}_s}\left(\frac{\partial P}{\partial q_s}\right)^r$$

where α is a positive constant and r is a rational number with odd numerator and denominator, $r \geqslant 1$. If $r \geqslant \lambda\mu/[2(\lambda-1)]$, then, for sufficiently small α, the function V is positive definite, and its derivative along trajectories of system (3.6) is negative definite.

Now consider a perturbed system, with the generalized forces in the form

$$Q_s=\frac{\partial}{\partial \dot{q}_s}\left(W(\dot{\mathbf{q}})+\sum_{j=1}^{k} b_j(t)W_j(\dot{\mathbf{q}})\right)$$

As in Examples 1 and 2, we shall assume that $W_j(\dot{\mathbf{q}})$ are continuously differentiable homogeneous functions of order σ, $\sigma > 2$, and that the functions $h_j(t)$ are continuous and bounded for $t \geqslant 0$ together with the integrals (3.4).

We construct a Lyapunov functions for the perturbed system in the form

$$V_1=V-\sigma\sum_{j=1}^{k}\int_0^t b_j(\tau)d\tau W_j(\dot{\mathbf{q}})-\alpha\sum_{s=1}^{n}\sum_{j=1}^{k}\int_0^t b_j(\tau)d\tau\left(\frac{\partial P}{\partial q_s}\right)^r\frac{\partial W_j}{\partial \dot{q}_s}$$

Checking whether the function V_1 satisfies the conditions of Lyapunov's asymptotic stability theorem, we see that if

$$\sigma > \max\{\mu/2+1;\ \mu-1+2/\lambda\}$$

the perturbations do not destroy the asymptotic stability of the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$.

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